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A STRONG CONTAINMENT PROPERTY FOR DISCRETE AMENABLE GROUPS OF AUTOMORPHISMS ON W^* ALGEBRAS

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This paper is dedicated with affection and gratitude to my father Dr. Jacob Granirer

ABSTRACT. Let G be a countable group of automorphisms on a W^* algebra $\mathcal{M}$ and let ϕ_0 be a w^*G_δ point of the set of G invariant states on $\mathcal{M}$ which belong to $w^*\,\mathrm{cl}\,\mathrm{Co}\,E$, where E is a set of (possibly pure) states on $\mathcal{M}$. If G is amenable, then the cyclic representation π_{ϕ_0} corresponding to ϕ_0 is *contained* in $\{\bigoplus \pi_\phi; \phi \in E\}$. This property characterizes amenable groups. Related results are obtained.

Introduction. Let $\mathcal{M}$ be an infinite dimensional W^* algebra, $\mathcal{M}^*$ its Banach space dual, G a group of automorphisms $g\colon \mathcal{M} \to \mathcal{M}$, and E a set of states on $\mathcal{M}$ such that $G^*E \subset E$. Denote by S_E^G the set of states ϕ in $w^*\,\mathrm{cl}\,\mathrm{Co}\,E$ (the w^* closure of the convex hull of E) which are G-invariant, i.e. $\phi(gx) = \phi(x)$ for all g in G and x in $\mathcal{M}$.

We call ϕ_0 in S_E^G a w^*G_δ point of S_E^G if there is a sequence $J = \{x_n\}$ in $\mathcal{M}$ such that $\{\phi_0\} = S_E^G \cap J^0$, where $J^0 = \{\phi \in \mathcal{M}^*; \langle \phi, J\rangle = 0\}$. This just means that the w^* topology *restricted* to S_E^G is first countable at ϕ_0. Denote by $w^*G_\delta(S_E^G)$ the set of all such points. We note that ϕ_0 being in $w^*G_\delta(S_E^G)$ depends only on the w^* topology *restricted to* S_E^G and does not depend a priori, on what happens in $w^*\,\mathrm{cl}\,\mathrm{Co}\,E$. Furthermore $w^*G_\delta(S_E^G)$ may be void. If for some countable $J \subset \mathcal{M}$ the set $(S_E^G \cap J^0, w^*)$ is separable metric or even if it has the RNP (even the WRNP is enough in some cases; see the sequel for notations), then $w^*G_\delta(S_E^G) \neq \varnothing$.

If $\phi \in \mathcal{M}^*$ is positive let π_ϕ be the GNS representation determined by ϕ (Pedersen [**12**, 3.3.3]).

The main result of this paper is (a refinement of) the following

THEOREM. *Let $\phi_0 \in w^*G_\delta(S_E^G)$. Then π_{ϕ_0} is contained (not only weakly contained à la Fell) in the direct sum $\{\bigoplus \pi_\phi; \phi \in E\}$ provided G is countable and amenable.*

If G is any nonamenable group then there is even an abelian $\mathcal{M} = L^\infty(X\mu)$ for some nonatomic probability space (X, B, μ) on which G acts ergodically and measure preservingly such that if E is the set of all pure states on $\mathcal{M}$ then $w^*G_\delta(S_E^G) = \{\mu\} = S_E^G$, yet π_μ is not contained (but only weakly contained) in $\{\bigoplus \pi_\phi; \phi \in E\}$.

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Several corollaries of this result follow and the unifying thread of them all is that they *characterize* the amenability of G in the class of countable groups. Furthermore they generalize to the noncommutative case our results in [**9**].

Corollary 1 deals with the case where E consists of *pure* states on $\mathcal{M}$. We show that in this case the "support" of any such $\phi_0 \in w^*G_\delta(S_E^G)$ is a countable set $R \subset \hat{\mathcal{M}}$ (the irreducible representations of $\mathcal{M}$ modulo unitary equivalence) such that $R = \bigcup R_k$, a disjoint union of *finite G invariant* subsets R_k, *provided G* is countable and *amenable*. Again in the absence of amenability this "w^*G_δ-finite invariant" property of points in $w^*G_\delta(S_E^G)$ fails (even for abelian $\mathcal{M}$). This corollary is a generalization to the nonabelian case of (an improved version of) our results in [**9**].

The remainder of the corollaries deal with the case that $S_E^G \cap J^0$ has the RNP (or S_E^G has the WRNP under the additional assumption that S^G is a simplex). In this case $w^*G_\delta(S_E^G) \neq \emptyset$.

One of the main tools in the proofs is the "w^*G_δ sequential property" of countable (left) amenable semigroups introduced in [**9**], a property which characterizes amenability in the class of countable groups. As shown in [**18**, pp. 47–48] there are noncountable abelian (a fortiori amenable) groups which do not possess this property.

In the end we point out that in case $E \subset \mathcal{M}_*$ consists of *normal* states better results are available and the amenability of G does not come into play.

Definitions and notations. $\mathcal{M}$ will *always* denote an infinite dimensional W^* *algebra*, $\mathcal{M}^*$ its Banach space dual (for the unexplained notations we follow Pedersen [**12**]), and $S_\mathcal{M} = S \subset \mathcal{M}^*$ the set of states. If $E \subset \mathcal{M}^*$, let $\operatorname{Co} E$ denote the convex hull of E and $w^* \operatorname{cl} \operatorname{Co} E$ the $w^* = \sigma(\mathcal{M}^*, \mathcal{M})$ closure of $\operatorname{Co} E$. (If X, Y are linear spaces in duality, $\sigma(X, Y)$ is the weakest topology on X which renders all linear functions in Y continuous.) $\mathcal{M}_*$ denotes the predual of $\mathcal{M}$.

If ϕ is a positive element in $\mathcal{M}^*$, then $(\pi_\phi, H_\phi, h_\phi)$ will denote the (GNS) cyclic representation induced by ϕ [**12**, 3.3.3]. It acts on the Hilbert space H_ϕ and is such that $\phi(x) = \langle \pi_\phi(x) h_\phi, h_\phi \rangle$, where $h_\phi \in H_\phi$ is the cyclic vector.

Let $\operatorname{Aut} \mathcal{M}$ be the set of all automorphisms of $\mathcal{M}$. Let $G \subset \operatorname{Aut} \mathcal{M}$ be a semigroup. If $g \in G$ let $g^*\colon \mathcal{M}^* \to \mathcal{M}^*$ be defined by $(g^*\phi)(x) = \phi(gx)$ for all x in $\mathcal{M}$.

If $E \subset \mathcal{M}^*$ let $G^*E = \{g^*\phi; g \in G,\ \phi \in E\}$. $G^*\phi = \phi$ will just mean that $g^*\phi = \phi$ for all g in G. If $E \subset S$ let $S_E^G = \{\phi \in w^* \operatorname{cl} \operatorname{Co} E; G^*\phi = \phi\}$ and $S^G = \{\phi \in S; G^*\phi = \phi\}$ the set of all G invariant states.

If $K \subset S$ then ϕ_0 is a w^*G_δ point of K if there are x_n in $\mathcal{M}$ and scalars α_n, $n = 1, 2, 3, \ldots$, depending on ϕ_0, such that $\{\phi_0\} = \{\phi \in K; \phi(x_n) = \alpha_n,\ n \geq 1\}$. $w^*G_\delta(K)$ denotes the set of all such ϕ in K. Since $\phi(I) = 1$ for all ϕ in S, $\phi_0 \in w^*G_\delta(K)$ iff there is a separable subspace $J \subset \mathcal{M}$ such that $J^0 \cap K = \{\phi_0\}$, where $J^0 = \{\phi \in \mathcal{M}^*; \phi(x) = 0 \text{ for all } x \text{ in } J\}$.

$\hat{\mathcal{M}}$ will denote the set of all irreducible unitary representations of $\mathcal{M}$ *modulo unitary* (spatial in [**12**, 3.3.6]) *equivalence*.

If (π_1, H_1), (π_2, H_2) are unitary representations of $\mathcal{M}$ [**12**, 3.3.1] we write $\pi_1 \leq \pi_2$ if π_1 is unitarily equivalent to a subrepresentation of π_2. If $g \in \operatorname{Aut} \mathcal{M}$ and (π, H) a representation of $\mathcal{M}$, then $\hat{g}\pi$ is the representation of $\mathcal{M}$ on H given by $(\hat{g}\pi)(x) = \pi(gx)$ i.e. $g\pi = \pi \circ g$. Note that if $(\pi_1 H_1)$, $(\pi_2 H_2)$ are unitarily equivalent representations (denoted by $\sim$), i.e. for some unitary $u\colon H_1 \to H_2$, $u\pi_2(x)u^* =$

$\pi_1(x)$ for all x then $u\pi_2(gx)u^* = \pi_1(gx)$ for all x. Thus $\hat{g}\pi_1, \hat{g}\pi_2$ are equivalent. Thus every g in Aut $\mathcal{M}$ acts on $\hat{\mathcal{M}}$ (where unitarily equivalent representations are identified).

If E is a set of pure states, denote by $\hat{E}$ the *subset of* $\hat{\mathcal{M}}$ given by $\hat{E} = \{\pi_\phi; \phi \in E\}^\wedge = \{\pi_\phi; \phi \in E\}/\sim$. If $g \in \text{Aut}\,\mathcal{M}$ and E is a set of *pure* states, then the equality $\hat{g}\hat{E} = \hat{E}$ will mean that $\{\pi_\phi; \phi \in E\}^\wedge = \{\pi_\phi \circ g; \phi \in E\}^\wedge$ (i.e. equality of sets in $\hat{\mathcal{M}}$).

If $\mathcal{M}$ is abelian then $\hat{\mathcal{M}}$ coincides with the set of all multiplicative states on $\mathcal{M}$. In general $\phi \to \pi_\phi$ will be a many-to-one map when ϕ ranges over the pure states of $\mathcal{M}$.

When no ambiguity arises we write equality when we mean $\sim$. $B(H)$ will denote the bounded linear operators on the Hilbert space H. The semigroup S is left (right) amenable if there is a left (right) invariant state (or mean) (see M. M. Day [**6**] for more on this topic). A convex set K of a Banach space X has the (weak) Radon-Nikodým property (WRNP) RNP if for any finite measure space $(XB\mu)$ any countably additive μ-continous map $m\colon B \to X$, such that $\mu(A)^{-1}m(A) \in K$ if $\mu(A) \neq 0$, is represented by a Bochner (Pettis) integrable function. For RNP (WRNP) sets see Stegall [**16**] (E. Saab [**15**]).

Main results.

THEOREM 1. *Let $\mathcal{M}$ be a W^* algebra and $G \subset \text{Aut}\,\mathcal{M}$ a group of automorphisms $g\colon \mathcal{M} \to \mathcal{M}$. Let E be a set of states on $\mathcal{M}$ such that $G^*E \subset E$ and denote $S_E^G = \{\phi \in w^*\,\text{cl}\,\text{Co}\,E; G^*\phi = \phi\}$.*

(a) *If G is countable and amenable and $\phi_0 \in w^*G_\delta(S_E^G)$ then there is a countable set $E_0 \subset E$ such that $\pi_{\phi_0} \leq \{\bigoplus \pi_\phi; \phi \in E_0\}$ and π_{ϕ_0}, π_ϕ are not disjoint* (*see Pedersen* [**12**, 3.8.12]) *for each ϕ in E_0.*

(b) *If S is any nonamenable group then there is some nonatomic probability space $(XB\mu)$ on which S acts ergodically as measure preserving transformations such that if E is the set of all pure states on $\mathcal{M} = L^\infty(X)$ then $S_E^G = w^*G_\delta(S_E^G) = \{\mu\}$, yet π_μ is not contained in $\{\bigoplus \pi_\phi; \phi \in E\}$.*

REMARKS. (i) Note that E need not be w^* closed but only G^* invariant.

(ii) It is enough in (a) that G be only a right amenable semigroup.

(iii) We show in (a) that there is some $h_0 \in \{\bigoplus H_\phi; \phi \in E_0\}$ such that $\phi_0(x) = \langle \pi(x)h_0, h_0\rangle$ for all x in $\mathcal{M}$, where $\pi = \{\bigoplus \pi_\phi; \phi \in E_0\}$.

PROOF. Let G be right amenable. Then the semigroup G^* is left amenable. By our Theorem 1 in [**9**] G^* has the w^*G_δ sequential property. Thus $\phi_0 = w^*\lim \phi_n$ where ϕ_n is a *sequence* in $\text{Co}\,E$. A result of Akemann, Dodds, and Gamelin [**1**] implies that even $w\lim_n \phi_n = \phi_0$, where $w = \sigma(\mathcal{M}^*, \mathcal{M}^{**})$ (while $w^* = \sigma(\mathcal{M}^*, \mathcal{M})$). It follows then that there exists a sequence ψ_n in $\text{Co}\{\phi_n\} \subset \text{Co}\,E$ such that $\|\psi_n - \phi_0\| \to 0$. Each ψ_n is a convex combination of a finite subset E_n of E. Let $E_0 = \bigcup_1^\infty E_n$. Thus if ϕ appears as a component in several ψ_n's, it appears in E_0 only once.

Let $\pi = \{\bigoplus \pi_\phi; \phi \in E_0\}$ where $(\pi_\phi, H_\phi, h_\phi)$ is the cyclic representation induced by ϕ. Each ψ_n is a *vector* state on the C^* *algebra* $\pi(\mathcal{M})$ (see [**12**, (1.5.7)]) acting on the Hilbert space $H = \{\bigoplus H_\phi; \phi \in E_0\}$, i.e. $\psi_n(x) = \langle \pi(x)g_n, g_n\rangle$ for some g_n in H (see Remark 1). Furthermore $\phi_0(x)$ is a *state* on the C^* algebra $\pi(\mathcal{M})$ since if $\pi x = 0$, then $\phi(x) = 0$ for all ϕ in E_0; hence $\psi_n(x) = 0$ for all n, and $\phi_0(x) = 0$. But we claim that $\|\psi_n - \phi_0\| \to 0$ in the norm of $\pi(\mathcal{M})$. This is readily implied by the fact that π is an *open* map since $\pi(\mathcal{M})$ is a C^* algebra [**12**, 1.5.7].

We now apply Theorem D of R. Kadison [**10**, p. 307] and get that ϕ_0 is also a *vector* state on $\pi(\mathcal{M})$ acting on H, i.e. there is some $h_0 \in H$ such that $\phi_0(x) = \langle \pi(x)h_0, h_0\rangle$.

If $\langle \pi_{\phi_0} H_{\phi_0} h_{\phi_0}\rangle$ is the cyclic corresponding to the state ϕ_0 on $\mathcal{M}$ then $\phi_0(x) = \langle \pi_{\phi_0}(x)h_{\phi_0}, h_{\phi_0}\rangle = \langle \pi(x)h_0, h_0\rangle$. It follows by Proposition 3.3.7 of [**12**] that π_{ϕ_0} is unitarily equivalent to π restricted to $[\pi(\mathcal{M})h_0] \subset H$. Hence $\pi_{\phi_0} \leq \{\bigoplus \pi_\phi; \phi \in E_0\}$. Let $E_0 = \{\eta_n; n = 1, 2, \ldots\}$ and $(\pi_n, H_n, h_n) = (\pi_{\eta_n}, H_{\eta_n}, h_{\eta_n})$. Then there are $v_n \in H_n$ such that $h_0 = \sum_h v_n$, $1 = \|h_0\|^2 = \sum_h \|v_n\|^2$, and $\phi_0(x) = \langle \pi(x)h_0, h_0\rangle = \sum_n \langle \pi(x)v_n, v_n\rangle = \sum_n \langle \pi_n(x)v_n, v_n\rangle$. Discard now from E_0 all η_n's for which the state $\langle \pi_n(x)v_n, v_n\rangle$ is 0 on $\pi_n(\mathcal{M})$ and let E_0 stand for the new set. If $\gamma_n(x) = \langle \pi_n(x)v_n, v_n\rangle = (\pi(x)v_n, v_n\rangle$ then $\gamma_n(x) \leq \phi_0(x)$ if $x \geq 0$ hence $\pi_{\gamma_n} \leq \pi_{\phi_0}$ by Pedersen [**12**, (3.3.8)] and $\pi_{\gamma_n} \leq \pi_{\eta_n}$ since π_{γ_n} is unitarily equivalent to π_n restricted to $[\pi_n(\mathcal{M})v_n]$ [**12**, 3.3.7]. This set E_0 will satisfy part (a) of the theorem.

(b) If G is any *nonamenable* group, there is by a result of Losert and Rindler [**11**] and J. Rosenblatt [**14**] a *nonatomic* probability space (X, B, μ) on which G acts ergodically as a group of measure preserving transformations $g\colon X \to X$ such that there exists a unique G-invariant state on $L^\infty(X)$, which is necessarily μ. Let E be the set of all pure states on $L^\infty(X) = \mathcal{M}$. Then $S_E^G = \{\mu\} = w^*G_\delta(S_E^G)$. Assume now that $\pi_\mu \leq \{\bigoplus \pi_\phi; \phi \in E\}$. Then, since each π_ϕ is one dimensional there are $\beta_n > 0$, $\sum \beta_n = 1$ such that $\mu(f) = \int f\, d\mu = \sum_{n=1}^\infty \beta_n \phi_n(f)$ for each f in $\mathcal{M}$ for some multiplicative ϕ_n in E. But then our argument on p. 112 of [**9**] shows that the measure μ has to contain atoms, which cannot be. □

REMARKS. (1) Let $\phi_1, \ldots, \phi_m$ be states on the C^* algebra A and $\psi = \sum_1^m \alpha_i \phi_i$ with $\alpha_i > 0$ and $\sum_1^m \alpha_i = 1$. Then $\pi_\psi \leq \bigoplus_1^m \pi_{\phi_i}$: let $(\pi_i, H_i, h_i) = (\pi_{\phi_i}, H_{\phi_i}, h_{\phi_i})$, then $\phi_i(x) = \langle \pi_i(x)h_i, h_i\rangle$. If $h = \sum_1^m \sqrt{\alpha_i} h_i$ then since $\pi(x)H_i \subset H_i$ we have

$$\begin{aligned} \langle \pi(x)h, h\rangle &= \sum_{i,j} \sqrt{\alpha_i \alpha_j} \langle \pi_i(x)h_i, h_j\rangle \\ &= \sum \alpha_i \langle \pi_i(x)h_i, h_i\rangle = \psi(x). \end{aligned}$$

Thus $\pi_\psi \leq \bigoplus_1^m \pi_i$.

(2) With the notations of part (a) of the theorem let E (hence $E_0 = \{\eta_n\}$) consist only of *pure* states. Thus $\pi_n = \pi_{\eta_n}$ are irreducible. Since $\gamma_n(x) = \langle \pi_n(x)v_n, v_n\rangle$ are nonzero and $\pi_{\gamma_n} \leq \pi_n$ we get that $\pi_n \leq \pi_{\phi_0} \leq \{\bigoplus \pi_k; k \geq 1\}$ for all n, where $\phi_0(x) = \sum \gamma_n(x)$ and $\sum \|v_n\|^2 = 1$.

Choose a maximal subsequence $\{\pi_{n_i}\} \subset \{\pi_n\}$ (possibly finite) such that π_{n_i}, π_{n_j} are not unitarily equivalent if $i \neq j$. If $\pi_n \sim \pi_{n_i}$ there is some $v_n^1 \in H_{n_i}$ such that $\gamma_n(x) = \langle \pi_{n_i}(x)v_n^1, v_n^1\rangle$ and $\|v_n\| = \|v_n^1\|$. Hence if $\varepsilon_i(x) = \{\sum \gamma_n(x); \pi_n \sim \pi_{n_i}\}$ then $\varepsilon_i(x) = \sum_k \langle \pi_{n_i}(x)v_k^i, v_k^i\rangle$ for some sequence $v_k^i \in H_{n_i}$ with $\sum_k \|v_k^i\|^2 < \infty$. Thus $\varepsilon_i \neq 0$ can be considered as a positive *normal* functional on $(\pi_{n_i}, H_{n_i} h_{n_i})$ and $\varepsilon_i(1) = \sum_k \|v_k^i\|^2 \neq 0$, $\phi_0(x) = \sum_i \varepsilon_i(x)$, and $\phi_0(1) = 1 = \sum_i \varepsilon_i(1)$.

Now let $Z_0 = \{z_\alpha; \alpha \in I\}$ be the set of all minimal central projections of the second dual $\mathcal{M}^{**}$ of $\mathcal{M}$.[1] For each g in G, g^{**} is an *automorphism* on $\mathcal{M}^{**}$ [**12**, (7.4.5), p. 244] and hence $g^{**}\colon Z_0 \to Z_0$, one-to-one onto. Since $z_\alpha z_\beta = 0$ unless $\alpha = \beta$, $(g^{**}z_\alpha)(g^{**}z_\beta) = 0$ unless $\alpha = \beta$ (see Dixmier [**7**, 5.2.4–5.2.8, p. 103]).

Let $z_{\alpha_i} \in Z_0$ be the central support of π_{n_i}. Then since ε_i is π_{n_i}-normal and is also in $\mathcal{M}^*$ we have $\varepsilon_i(xz_{\alpha_i}) = \varepsilon_i(x)$ for all x in $\mathcal{M}$ (where we consider $\mathcal{M} \subset \mathcal{M}^{**}$) and $\varepsilon_i(xz_\alpha) = 0$ if $\alpha \in I$ and $\alpha \neq \alpha_i$ for all x in $\mathcal{M}$ (see Takesaki [**17**, p. 125] and Akemann and Shulz [**2**, Proposition A1, p. 110 and Proposition A10, p. 116]). Now denote $\beta_j = \varepsilon_j(1) = \varepsilon_j(z_{\alpha_j})$ and fix some β_i. Let $F = \{\alpha_j; \beta_j = \beta_i\}$. Clearly F is finite since $\sum \beta_j = 1$. Let $g \in G$, then $g^{**}z_{\alpha_i} = z_\delta$ for some δ in I. But

$$\phi_0(z_{\alpha_i}) = \sum_j \varepsilon_j(z_{\alpha_i}) = \varepsilon_i(z_{\alpha_i}) = \beta_i = \phi_0(g^{**}z_{\alpha_i}) = \phi_0(z_\delta) = \sum_j \varepsilon_j(z_\delta).$$

But $\varepsilon_j(z_\delta) \neq 0$ for at most one j and since $\beta_i \neq 0$ there is such a j (see [**7**, 5.2.8]). Thus $g^{**}z_\delta = z_{\alpha_k}$, $\varepsilon_k(z_{\alpha_k}) = \beta_k = \beta_i$ for some k, and thus $\alpha_k \in F$. If $Z_F = \{z_{\alpha_j}; \alpha_j \in F\}$ then we have shown that $g^{**}Z_F \subset Z_F$ and since Z_F is finite and g^{**} is one-to-one, $g^{**}Z_F = Z_F$ for all g in G. This however implies that $\{\pi_{n_j} \circ g; \alpha_j \in F\} = \{\pi_{n_j}; \alpha_j \in F\}$ for all g, since π_{n_j} and $\pi_{n_j} \circ g$ are irreducible (see [**12**, (3.8.12), (3.13.3)]).

Now let β_{i_k} be a maximal set of different β_i's. For each k let $F_k = \{\alpha_j; \beta_j = \beta_{i_k}\}$.

Let $R_k = \{\pi_{n_j}; \beta_j = \beta_{i_k}\}$. Then the R_k are finite, pairwise disjoint sets such that $\bigcup_k R_k = \{\pi_{n_i};\ i = 1, 2, \ldots\}$. Furthermore for each g in G and k, $\{\pi_{n_j} \circ g; \pi_{n_j} \in R_k\} = R_k$ (up to unitary equivalence). Each finite set R_k can be further partitioned into finitely many *minimal* G-invariant sets R_j^k, i.e. subsets such that $\{\rho \circ g; g \in G\} = R_j^k$ for each ρ in R_j^k and $\{\rho \circ g; \rho \in R_j^k\} = R_j^k$ for each g in G (all above equalities are up to unitary equivalence). We thus have

COROLLARY 1. (a) *Let G be a group of automorphisms on the W^* algebra $\mathcal{M}$ and E a set of pure states on $\mathcal{M}$ such that $G^*E \subset E$ and let $S_E^G = \{\phi \in w^* \operatorname{cl} \operatorname{Co} E; G^*\phi = \phi\}$. Assume that $\phi_0 \in w^*G_\delta(S_E^G)$.*

If G is countable and amenable then there is a countable subset $E_0 \subset E$ such that

$$\pi_\psi \leq \pi_{\phi_0} \leq \left\{\bigoplus \pi_\phi; \phi \in E_0\right\} \quad \textit{for all } \psi \in E_0 \ (\textit{by Theorem } 1(\text{a})).$$

Furthermore $\{\pi_\phi; \phi \in E_0\}^\wedge = \hat{E}_0 = \bigcup_k R_k$ is a countable (or finite) disjoint union of finite minimal G invariant sets $R_k \subset \hat{\mathcal{M}}$, i.e. the finite sets R_k satisfy $R_k \cap R_j = \varnothing$ if $k \neq j$, $\hat{g}R_k = R_k$, and $\{\rho \circ g; g \in G\} = R_k$ for all g in G, ρ in R_k and all k.

(b) *No nonamenable group has the above "strong containment property" as part* (b) *of the previous theorem shows.*

REMARKS. (1) G need only be a right amenable semigroup (as in Day [**6**]).

(2) Fix k and define in G the equivalence relation $g_1 \sim g_2$ iff $\rho \circ g_1 = \rho \circ g_2$ (are unitarily equivalent) for all ρ in R_k. Then G modulo $\sim$ becomes a semigroup G_k of one-to-one maps on the finite set R_k. Thus G_k is a finite group and, if G

[1] We acknowledge with thanks communications we had with Alan L. T. Paterson. The proof below and the statement of Corollary 1 are different than the ones suggested in these inspiring communications.

is a group, for each k, card R_k is the cardinality of a finite coset G/H_k for some subgroup $H_k \subset G$.

If $J \subset \mathcal{M}$ denote $J^0 = \{\phi \in \mathcal{M}^*; \langle \phi, f \rangle = 0 \text{ for all } f \text{ in } J\}$.

COROLLARY 2. *Let $G \subset \operatorname{Aut} \mathcal{M}$ be a semigroup and E a set of pure states on $\mathcal{M}$. Assume that for some countable $J \subset \mathcal{M}$, $S_E^G \cap J^0$ is nonvoid and has the RNP.*

(a) *If G is countable and amenable then every subset $E_1 \subset w^* \operatorname{cl} E$ such that $G^* E_1 \subset E_1$ and $S_{E_1}^G \cap J^0 \neq \varnothing$ contains a finite subset $E_0 \subset E_1$ such that $\hat{g}\hat{E}_0 = \hat{E}_0$ for all g in G and $\{\pi_\phi \circ g; g \in G\} = \hat{E}_0$ for all ϕ in E_0 (i.e. $\hat{E}_0$ is a finite minimal G invariant subset of $\hat{E}_1$).*

(b) *If G is any nonamenable group then the ergodic measure preserving action of G on $L^\infty(X) = \mathcal{M}$, for the nonatomic probability space $(XB\mu)$ of Theorem 1(b), satisfies $S^G = \{\mu\}$ and has the RNP, yet for no finite set E_0 of pure states on $\mathcal{M}$ does $\hat{g}\hat{E}_0 \subset \hat{E}_0$, for all g in G (equivalently $G^* E_0 \subset E_0$), hold true.*

REMARKS. (i) w compact or norm separable w^* compact convex sets have the RNP (Stegall [**16**, Proposition 1.10]). (ii) G need only be right amenable. (iii) $S_{E_1}^G \neq \varnothing$ by the Markov-Kakutani-Day fixed point theorem, however $S_{E_1}^G \cap J^0 \neq \varnothing$ may not hold, hence we need to assume it.

PROOF. (a) If $F = w^* \operatorname{cl} E$ then $S_E^G = S_F^G$; hence we can assume that E is w^* closed. Now subsets of closed bounded RNP sets have the RNP [**16**, p. 508]; hence $S_{E_1}^G \cap J^0 \neq \varnothing$ has the RNP. Since $S_{E_1}^G \cap J^0$ is w^* compact and has the RNP it necessarily has a $w^* G_\delta$ point ϕ_0 (see [**16**] or for alternate proof see [**9**, p. 116]) which is necessarily a $w^* G_\delta$ point of S_E^G, since J is countable. Corollary 1(a) finishes the proof.

(b) The easy proof is left for the reader (or see remark (a) on p. 115 of [**9**]). □

If we assume that $\mathcal{M}$ is abelian then we get the following improvement (in a sense) of our Theorem 4 of [**9**]:

COROLLARY 3. (a) *Let $\mathcal{M}$ be an abelian W^* algebra, i.e. $\mathcal{M} = L^\infty(\Gamma, \mu)$ for some locally compact Γ and positive Radon measure μ [**17**, Theorem 1.18, p. 119].*

Let E be a set of multiplicative states on $\mathcal{M}$ and $G \subset \operatorname{Aut} \mathcal{M}$ a countable (right) amenable semigroup such that the nonvoid set $S_E^G \cap J^0$ has the RNP for some countable $J \subset \mathcal{M}$. If $E_1 \subset w^* \operatorname{cl} E$ is any set such that $G^* E_1 \subset E_1$ and $S_{E_1}^G \cap J^0 \neq \varnothing$ then E_1 contains a finite subset E_0 such that $G^* E_0 \subset E_0$.

(b) *No nonamenable group has the above "RNP-finite invariant property" (by Corollary 2(b)).*

REMARKS. (1) If $\phi_0 \in S_E^G \cap J^0$ then the set $F = \operatorname{supp} \phi_0$ is a w^* closed subset of E such that $G^* F \subset F$ (where $\operatorname{supp} \phi_0$ is the smallest w^* closed set $E' \subset E$ such that $\phi_0 \in w^* \operatorname{cl} \operatorname{Co} E'$).

(2) If $J = \{0\}$ then S_E^G need only have WRNP in order that Corollary 3(a) hold (see [**9**, Corollary 6]). We improve this in the next corollary.

If we take $J = \{0\}$ and impose certain restrictions on E and on the action of G on $\mathcal{M}$ then we can replace the RNP by the WRNP:

COROLLARY 4. (a) *Let $G \subset \operatorname{Aut} \mathcal{M}$ be a group such that S^G is a simplex. Let E be a set of pure states on $\mathcal{M}$ such that $u^* \phi u \in E$ for each unitary u in $\mathcal{M}$ and $\phi \in E$. Assume that $S_E^G \neq \varnothing$ and has the WRNP.*

If G is countable and amenable then every set $E_1 \subset w^* \operatorname{cl} E$ such that $G^* E_1 \subset E_1$ contains a finite subset E_0 such that $\hat{g}\hat{E}_0 = \hat{E}_0$ and $\{\hat{g}\pi_\phi; g \in G\} = \hat{E}_0$ for all g in G and ϕ in E_0 (i.e. such that $\hat{E}_0$ is finite minimal G invariant).

(b) *No nonamenable group has the above property (by Corollary* 2(b)).

PROOF. Since S^G is a simplex the positive cone $C^G = \{\bigcup \lambda S^G; \lambda \geq 0\}$ is a lattice, i.e. $\phi_1, \phi_2 \in C^G$ implies that $\phi_1 \wedge \phi_2$ exists and is in C^G (see Asimov and Ellis [**3**, pp. 49, 70]). We claim that $C_E^G = \{\bigcup \lambda S_E^G; \lambda \geq 0\}$ is also a lattice. In fact let $\phi_1, \phi_2 \in C_E^G$ and $\phi_0 = \phi_1 \wedge \phi_2 \in C^G$. We show that $\phi_0 \in C_E^G$. Clearly $\phi_0 \leq \phi_1$, hence by [**12**, 3.3.8], $\pi_{\phi_0} \leq \pi_{\phi_1}$, i.e. $\phi_0(x) = \langle \pi_{\phi_1}(x)\xi_1, \xi_1 \rangle$ for some $\xi_1 \in H_{\phi_1}$. Now for all $x, \phi_1(x) = \lim \phi_\alpha(x)$ where $\phi_\alpha \in \operatorname{Co} \lambda_1 E$ where $\lambda_1 = \|\phi_1\|$. Also, if $\pi_\mu(x) = 0$ for all $\mu \in E$ then $\pi_\mu(x^*x) = 0$ for all $\mu \in E$; thus $\mu(x^*x) = 0$ for all $\mu \in E$ hence $\phi_1(x^*x) = 0$ which implies, since $\phi_0 \leq \phi_1$, that $\phi_0(x^*x) = 0$. Thus by Cauchy-Schwarz, $\phi_0(x) = 0$. Hence $\phi_0 = 0$ on $\{\bigcap \operatorname{Ker} \pi_\mu; \mu \in E\}$. If $\phi_0 = 0$ then $\phi_0 \in C_E^G$. If $\phi_0 \neq 0$ we can assume that ϕ_0 is a state. Then ϕ_0 is a w^* limit of positive elements of type $\sum_1^n \langle \pi_{\mu_i}(x)\xi_i, \xi_i \rangle = \nu(x)$ with $\nu(1) = 1$ for some ξ_i in H_{μ_i} and μ_i in E, by Dixmier [**7**, 3.4.2 and 3.4.4, p. 66].

We now claim that $x \to \langle \pi_\mu(x)\xi_0\xi_0 \rangle = \mu_0(x)$ belongs to $w^* \operatorname{cl} E$ for each $\mu \in E$ and $\xi_0 \in H_\mu$ such that $\|\xi_0\| = 1$. In fact there is a unitary $u \in B(H_\mu)$ such that $u\xi_\mu = \xi_0$ and there is a net of unitaries $u_\alpha \in \mathcal{M}$ such that $\pi_\mu(u_\alpha) \to u$ strongly on H_μ by Kaplanski's density theorem [**12**, 2.3.3] and since π_μ is irreducible. But then $\pi_\mu(x)\pi_\mu(u_\alpha)\xi_\mu \to \pi_\mu(x)\xi_0$ in norm. Thus for all x in $\mathcal{M}$, $\langle \pi_\mu(u_\alpha^* x u_\alpha)\xi_\mu, \xi_\mu \rangle \to \langle \pi_\mu(x)\xi_0, \xi_0 \rangle$ as readily seen. (If $\|\xi_\alpha - \xi\| \to 0$ then for all bounded T, $\langle T\xi_\alpha, \xi_\alpha \rangle \to \langle T\xi, \xi \rangle$.) It follows then that ϕ_0 is a w^* limit of a net in $\operatorname{Co}\{w^* \operatorname{cl} E\}$ hence $\phi_0 \in S^G \cap w^* \operatorname{cl} \operatorname{Co} E$ which by definition is S_E^G. We have shown that for all $\phi_1, \phi_2 \in C_E^G$, $\phi_1 \wedge \phi_2$ exists and is in C_E^G. But then $C_E^G - C_E^G$ (the linear span of C_E^G) is a lattice [**3**, p. 49]). But then $C_E^G - G_E^G$, over the reals, is an abstract L space [**3**, p. 70, Theorem 7.1 and p. 14, Theorem 4.7]. Now $S_E^G - S_E^G$ has the WRNP by Saab [**15**, Theorem 1(i)]. However this last set is just the closed unit ball of the abstract L space $C_E^G - C_E^G$. Thus the Banach lattice $C_E^G - C_E^G$ has the WRNP. We now apply Proposition 8 of Ghoussoub and Saab [**8**] and get that $C_E^G - C_E^G$ even has the RNP. and hence so does (each bounded subset) S_E^G. Now apply Corollary 2(a) with $J = \{0\}$. □

REMARKS. S^G is a simplex at least in the case when $(\mathcal{M}, G, \alpha)$ is weakly asymptotically abelian (Pedersen [**12**, 7.13.1] or even if α_G is a large group of automorphisms of $\mathcal{M}$ [**12**, 7.12.5 and 7.13.2]). However S^G may be a simplex, yet α_G need not be a large group of automorphisms (Takesaki [**17**, pp. 252–253]). S^G is a simplex if and only if the pair $(\mathcal{M}, \phi)$ is G-abelian (Bratteli-Robinson [**4**, Definition 4.3.6, p. 374]) for each $\phi \in S^G$, iff $E_\phi \pi_\phi(\mathcal{M}) E_\phi$ is abelian for all $\phi \in S^G$ where E_ϕ is the projection from H_ϕ to $\{\xi \in H_\phi; u_\phi(g)\xi = \xi$ for all g in $G\}$ (see [**4**, Corollary 4.3.11, p. 379]).

THE CASE WHERE $E \subset \mathcal{M}_*$. If E is a set of *normal* states on $\mathcal{M}$ then G need not be amenable and $G^*E \subset E$ need not hold in Theorem 1 and its corollaries. Stronger results are available in this case by adapting the results in [**19**] to the W^* algebra context. If $K \subset \mathcal{M}^*$, $w^* \operatorname{seq} \operatorname{cl} K$ denotes the w^* sequential closure of K. $\mathcal{M}_*$ is considered as embedded in $\mathcal{M}^*$.

THEOREM 2. *Let $\mathcal{M}$ be a W^* algebra and $G \subset \operatorname{Aut}\mathcal{M}$ a countable set. Let $E \subset \mathcal{M}_*$ be a set of (normal) states and $S_E^G = \{\phi \in w^* \operatorname{cl}\operatorname{Co} E; G^*\phi = \phi\}$.*

If $\phi_0 \in w^*G_\delta(S_E^G)$ then $\phi_0 \in \operatorname{norm}\operatorname{cl}\operatorname{Co} E \subset \mathcal{M}_*$ and $\pi_{\phi_0} \leq \{\bigoplus \pi_\phi; \phi \in E_0\}$ for some countable set $E_0 \subset E$.

Furthermore, if for some countable $J \subset \mathcal{M}$, $S_E^G \cap J^0$ has the WRNP or $\operatorname{card} S_E^G \cap J^0 < 2^c$ and in fact if $S_E^G \cap J^0$ does not contain a "w^* affine isomorph" of the "big" set $\mathcal{F} = \{\phi \in S_{l^\infty}; \phi(f) = 0$ for f in $c_0\} \subset (l^\infty)^*$ then there is some ϕ_0 in $S_E^G \cap J^0 \cap \operatorname{norm}\operatorname{cl}\operatorname{Co} E \subset \mathcal{M}_*$ such that $\pi_{\phi_0} \leq \{\bigoplus \pi_\phi; \phi \in E_0\}$ for some countable $E_0 \subset E$.

REMARKS. $c_0 = \{f \in l^\infty; \lim_n f(n) = 0\}$ and S_{l^∞} is the set of states on l^∞. By "w^* affine isomorph" we mean by a w^*-w^* continuous norm isomorphism into, $t^*: l^{\infty *} \to \mathcal{M}^*$, such that $t^*\mathcal{F} \subset S_E^G \cap J^0$, as in [**19**, Theorem 1.4(b)]. Note that $\mathcal{F} = w^* \operatorname{cl}\operatorname{Co}(\beta N \sim N)$ and $\operatorname{card}\mathcal{F} = 2^c$, where c is the cardinality of the continuum, as well known. G can be replaced by any set of w^* continuous operators on $\mathcal{M}$. This theorem is false for uncountable G.

PROOF. Any g in $\operatorname{Aut}\mathcal{M}$ is w^* continuous on $\mathcal{M}$ [**17**, p. 135, Corollary 3.10]. It follows now that $\{\phi_0\} \subset w^* \operatorname{seq}\operatorname{cl}\operatorname{Co} E$ since if not then by Theorem 1.4(b) of [**19**] the set $\{\phi_0\}$ would contain a w^* affine isomorph of $\mathcal{F}$, which cannot be. (We cannot apply our Theorem 1 here since G need not be amenable.) Thus $\phi_0 = w^* \lim \phi_n$ for some sequence ϕ_n in $\operatorname{Co} E$. The rest of the proof is verbatim like that of Theorem 1.

If $S_E^G \cap J^0$ has the WRNP or $\operatorname{card}(S_E^G \cap J^0) < 2^c$ then apply Corollary 1.4′ of [**19**] and get that there is some ϕ_0 in $S_E^G \cap J^0 \cap w^* \operatorname{seq}\operatorname{cl}\operatorname{Co} E$. But then, the proof of Theorem 1 shows that $\phi_0 \in \operatorname{norm}\operatorname{cl}\operatorname{Co} E$ and $\pi_{\phi_0} \leq \{\bigoplus \pi_\phi; \phi \in E_0\}$ for some countable set $E_0 \subset E$.

If $S_E^G \cap J^0$ does not contain a "w^* affine isomorph" of $\mathcal{F}$ then apply Theorem 1.4(b) of [**19**] and get that there is some $\phi_0 \in S_E^G \cap J^0 \cap w^* \operatorname{seq}\operatorname{cl}\operatorname{Co} E$. For the rest argue as above.

ADDED IN PROOF. We have obtained the following result using results in [**19**] (which in turn use in part techniques of Ching Chou):

THEOREM. *Let $G \subset \operatorname{Aut}\mathcal{M}$ be a countable group and E a set of pure states on $\mathcal{M}$ such that $G^*E \subset E$ and $J \subset \mathcal{M}$ countable.*

(a) *Assume that G is amenable*: (i) *If $\phi \neq S_E^G \cap J^0 \subset \mathcal{M}_*$ then $\mathcal{M}$ contains minimal projections (in the absence of such, $S^G \cap J^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}$ contains a "w^* affine isomorph of the big set $\mathcal{F}$").* (ii) *$S^G \cap \mathcal{M}_*^\perp$ contains a "w^* isomorph of the big set $\mathcal{F}$").*

A fortiori no action of a countable amenable group on an infinite dimensional W^* algebra has a unique G invariant state on $\mathcal{M}$.

(b) *If G is any nonamenable group then the action of G on $L^\infty(X)$ of Theorem* 1(b) *violates both* (a)(i) *and* (a)(ii).

REMARKS. 1. $\mathcal{M}^* \sim \mathcal{M}_*$ is the set theoretical difference of $\mathcal{M}^*$ and $\mathcal{M}_* \subset \mathcal{M}^*$. $\mathcal{M}_*^\perp$ is the set of singular elements of $\mathcal{M}^*$ [**17**, p. 127].

2. This theorem improves our Theorem 3 in [**9**] which in turn is a result of K. Schmidt [**20**] and J. Rosenblatt [**14**]. Our only assumption on $\mathcal{M}$ is that it is infinite dimensional.

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